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ScienceDirect
Journal of Applied Mathematics and Mechanics 71 (2007) 415–423
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

The equilibrium shapes of non-uniform vortices in the ocean☆

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Received 27 January 2006

Abstract

The problem of the equilibrium shapes of rotating vortices in a stratified ocean, which is in a statically stable state, is considered within the framework of the model of an ideal incompressible fluid. An equilibrium shape is a surface on which the pressures in the vortex and in the ocean are equal, in which case, on this surface the normal components of the velocities of the media are equal to zero, but a discontinuity of the tangential components is allowed. The case of stratification of the media along the local vertical is considered. The external medium (the ocean) may consist of several layers, differing sharply in density.

The problem of the longevity of vortices in the World ocean, having various shapes (in oceanology deep-water vortices are called "lenses", and shallow-water vortices are called "rings"), is obviously related to the presence of equilibrium shapes of a rotating liquid in a stratified continuous medium, which is in static equilibrium. This problem is considered in different formulations, in particular, in Refs. 1–4, where a procedure for the construction and the form of the equilibrium shapes in the case of a uniform density of the vortices and linear stratification of the ocean surrounding them are described. Below we consider other approaches to constructing the equilibrium shapes, which enable the problem of constructing the shapes in certain cases of non-uniform density of the vortices and also in the case of an arbitrarily varying density of the external medium along the vertical to be solved.

1. Formulation of the problem

Equilibrium shapes of a uniform "lens", the centre of which is fixed, and which is rotating in an unperturbed linearly stratified ocean, were constructed in Refs. 1–3. Below it is assumed that the density of the lens and of the ocean are known functions of only one vertical coordinate z.

We will consider Euler's equations in the Gromeka Lamb form[5]

$$[(2\mathbf{\Omega} + \mathrm{rot}\,\mathbf{v}) \times \mathbf{v}] = -\rho^{-1}\mathrm{grad}\,p - \mathrm{grad}(V^2/2 + gz); \quad V = |\mathbf{v}| \tag{1.1}$$

written for the steady case in axes $Cxyz$ connected with the rotating Earth on the assumption that the force of gravity has constant components over the whole lens, i.e., $g = \mathrm{const}$. The point C is the conventional centre of the lens, at rest in the ocean; the x, y and z axes are directed east, north and along the local normal (upwards) at the point C, Ω is the angular velocity of rotation of the Earth, v is the velocity vector, p is the pressure, and $\rho \equiv \rho(z)$ is the density of the

☆ *Prikl. Mat. Mekh*. Vol. 71, No. 3, pp. 458–467, 2007.
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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2007.07.007

liquid of which the lens consists. Assuming that the vertical components of the velocity in the body of the lens are zero ($\upsilon_z = 0$), we can write Eq. (1.1) in the scalar form

$$\Phi_x \equiv \rho(\upsilon_x \partial\upsilon_x/\partial x + \upsilon_y \partial\upsilon_x/\partial y - 2\Omega_z \upsilon_y) = -\partial p/\partial x$$
$$\Phi_y \equiv \rho(\upsilon_x \partial\upsilon_y/\partial x + \upsilon_y \partial\upsilon_y/\partial y + 2\Omega_z \upsilon_x) = -\partial p/\partial y \qquad (1.2)$$
$$\Phi_z \equiv \rho(g - 2\upsilon_x \Omega_y) = -\partial p/\partial z$$

The components of the angular velocity of rotation of the Earth in these formulae Ω_y, Ω_z are constant quantities and $\Omega_x = 0$. We will supplement relations (1.2) with the continuity equation, which in the case considered has the form

$$\partial\upsilon_x/\partial x + \partial\upsilon_y/\partial y = 0 \qquad (1.3)$$

Suppose the horizontal components of the velocity are related to x and y by the linear low

$$\upsilon_x = \alpha(z)x + a_y(z)y + a_z(z), \quad \upsilon_y = -\alpha(z)y + b_x(z)x + b_z(z) \qquad (1.4)$$

with coefficients, which are as yet unknown functions, and $a_z(0) = b_z(0) = 0$. The densities of the lens and the ocean (each in its own region), and all the functions occurring in expansion (1.4), as well as the required equilibrium surface will be assumed initially to be continuously differentiable; in particular, we will assume that this surface has no edges or other similar singularities. For the velocity field, defined by formulae (1.4), the continuity Eq. (1.3) is satisfied automatically.

We will obtain the conditions of compatibility of Eq. (1.2) (from the criterion for the existence of a total differential in the differential expression $\Phi_x dx + \Phi_y dy + \Phi_z dz \equiv dp(x, y, z)$). Using equalities (1.4) we obtain the components of the vector Φ. The compatibility condition $\partial\Phi_x/\partial y = \partial\Phi_y/\partial x$ is satisfied automatically. For the other two conditions ($\partial\Phi_x/\partial z = \partial\Phi_z/\partial x$ and $\partial\Phi_y/\partial z = \partial\Phi_z/\partial y$), equating terms with the variables x and y for the first and zero powers, we arrive at a system of equations in five functions of the variable z: α, a_x, a_y, b_x and b_y (henceforth we will use a prime to denote derivatives with respect to the z coordinate)

$$\rho'[\alpha^2 + b_x(a_y - 2\Omega_z)] + \rho[2\alpha\alpha' + b_x'(a_y - 2\Omega_z) + b_x a_y'] = 0$$
$$\rho'[\alpha a_z + b_z(a_y - 2\Omega_z)] + \rho[\alpha(a_z' + 2\Omega_y) + b_z'(a_y - 2\Omega_z) + \alpha' a_z + a_y' b_z] = 0$$
$$2\Omega_z(\rho'\alpha + \rho\alpha') = 0 \qquad (1.5)$$
$$\rho'[\alpha^2 + a_y(b_x + 2\Omega_z)] + \rho[2\alpha\alpha' + a_y'(b_x + 2\Omega_z) + a_y b_x'] = 0$$
$$\rho'[a_z(b_x + 2\Omega_z) - \alpha b_z] + \rho[a_z'(b_x + 2\Omega_z) - \alpha b_z' + 2a_y\Omega_y + b_x' a_z - \alpha' b_z] = 0$$

Equalities (1.5) guarantee that the compatibility conditions are satisfied identically.

System (1.5) has the following three first integrals

$$\Omega_z\alpha\rho = c_1, \quad \rho(\alpha^2 + b_x a_y - 2\Omega_z b_x) = c_2, \quad \rho(\alpha^2 + b_x a_y + 2\Omega_z a_y) = c_3 \qquad (1.6)$$

Suppose the constant c_1, c_2 and c_3 of these integrals are known. Then, the unknown functions of the model (1.4) can be found: they are determined in terms of the final formulae or in quadratures.

The equilibrium shape of a lens must satisfy the condition that no particles either of the lens itself or of the ocean can pass through its surface (i.e. the condition for the normal projections of the velocities onto the boundary of the lens to be equal to zero). We will assume,[1–3] that the ocean outside the lens is stratified and is in static equilibrium. The density of the ocean ρ_f and the pressure p_f are certain functions of the vertical coordinate z. The following equality must hold on the surface of the equilibrium shape

$$p(x, y, z) - p_f(z) = 0 \qquad (1.7)$$

The normal $\mathbf{n}$ at each point of this surface is collinear with the vector grad$(p - p_f(z))$. We will express grad p from the fundamental Eq. (1.1) in terms of the kinematic variables. Since the velocity field (1.4) considered assumes that the vertical component of the velocity v_z is equal to zero, it is sufficient to consider the impermeability condition $\mathbf{v}\cdot\mathbf{n}=0$, and consequently, the vector $\mathbf{n}$ only along the horizontal coordinates x and y. We have

$$\mathbf{n} = -\rho(z)\{[(\mathrm{rot}\mathbf{v}+2\mathbf{\Omega})\times\mathbf{v}]+\mathrm{grad}(V^2/2)\}$$

Then, we have the following condition on the lens surface

$$\frac{\partial\upsilon_x}{\partial x}(\upsilon_x^2-\upsilon_y^2)+\upsilon_x\upsilon_y\left(\frac{\partial\upsilon_y}{\partial x}+\frac{\partial\upsilon_x}{\partial y}\right) = 0 \tag{1.8}$$

or, taking relations (1.3) and (1.4) into account,

$$\alpha(\upsilon_x^2-\upsilon_y^2)+(b_x+a_y)\upsilon_x\upsilon_y = 0 \tag{1.9}$$

2. A uniform lens

It can be shown that a lens situated at the equator ($\Omega_z=0$), cannot have an equilibrium shape even when ρ is not constant. We will therefore consider the general case when $\Omega_z\neq 0$. Suppose the lens has constant density $\rho(z)=\rho_0=\mathrm{const}$. According to conditions (1.5), α, a_y and b_x are constant quantities, while the function a_z and b_z are linear in z. Suppose the ocean has a linear stratification[1–3]

$$p_f = p_f(0)-\rho_0 gz-(gz^2/2)\partial\rho_f/\partial z\big|_{z=0}$$

As follows from this formula, and also from Eq. (1.2), the surface (1.7) is a second-order surface in the variables x, y, z. If condition (1.9) is not satisfied identically, the left-hand side of it is also a quadratic function $F(x, y, z)$, in view of relations (1.4). On the lens surface $p-p_f=0$ the function F vanishes. Hence we conclude that the conditions $F(x, y, z)=0$ and $p-p_f=0$ describe the same second-order surface. Otherwise (condition (1.9) is satisfied identically), the function F is identically equal to zero and the following equalities are satisfied

$$\alpha = a_y+b_x = 0 \tag{2.1}$$

Nevertheless condition (2.1) cannot not be satisfied. In fact, when $v=0$ condition (1.9) is automatically satisfied. But conditions (1.9) and (1.7), which specify the equilibrium shape, are equivalent. Hence we conclude that the centre of rotation in each section z lies on the surface of the lens. This conclusion, however, contradicts common sense, and hence the only way to avoid a contradiction is to assume that Eq. (2.1) are satisfied.

We will consider the functions a_z and b_z, which are linear in z by virtue of system (1.5): $a_z=\chi z$ and $b_z=\kappa z$, where the constants κ and χ must satisfy the system

$$\kappa(a_y-2\Omega_z) = 0,\quad \chi(b_x+2\Omega_z)+2a_y\Omega_y = 0 \tag{2.2}$$

If, in the first of these relations, the expression in brackets is equal to zero, i.e. $a_y=2\,\Omega_z$, then $b_x+2\,\Omega_z=0$ by virtue of Eq. (2.1). If $a_y\neq 0$, then $\Omega_y=0$; in the opposite case, $a_y=\Omega_z=0$, i.e., the lens is situated either at the Earth's pole or at the equator. If it is situated at the pole ($\Omega_y=0$), we have a special case, since the Earth rotates in a positive direction around the z axis with angular velocity $\Omega_z=\Omega$, while the lens rotates in the opposite direction with double this velocity. It is easy to show that the equilibrium shape here is indeterminate, in the same way as the parameters χ and κ are not defined uniquely from relations (2.2). The reason for this paradoxical situation is the equality of the oppositely directed centrifugal force (due to the inherent rotation of the lens) and of the Coriolis force of inertia; their sum is always equal to zero.

In the general case when $a_y-2\Omega_z\neq 0$ (this, in particular, can again be the Earth's pole), taking relations (2.1) and (2.2) into account we arrive at the equalities

$$\chi = \frac{2\Omega_y b_x}{b_x+2\Omega_z},\quad \kappa = 0$$

i.e., $v_x=-b_x y+\chi z$, $v_y=b_x x$. Apart from the notation, we have the same formulae which specify the velocity field inside the lens as in Refs. 1–3. In this case, the parameter b_x has the meaning of the constant angular velocity of rotation of the lens ω_0, while the ratio $\chi/b_x = 2\Omega_y/(b_x + 2\Omega_z)$ represents the position of the centre of rotation of the lens ($x_0 = 0$, $y_0 = (\chi/b_x)z$) in each horizontal section $z = \text{const}$. The geometrical position of all the centres of rotation is a straight line in the (y, z) plane, which is inclined at a small angle to the y axis (to the z axis) for values of the angular velocity ω_0 that are small (large) in modulus. For linear stratification of the ocean the equilibrium shape of the lens is some second-order surface.

3. A non-uniform lens

In the case of an arbitrary function $\rho = \rho(z)$, the solution of the problem will also be constructed on the assumption that equalities (2.1) are satisfied (in fact the need for them to be satisfied can be strictly validated). The impermeability condition (1.9) is then also satisfied.

It follows from the last two relations of (1.6) that $\rho b_x(b_x + 2\Omega_z) = c$, i.e.,

$$-a_y = b_x = -\Omega_z \pm \sqrt{(\Omega_z^2 + c/\rho)} \tag{3.1}$$

and, by virtue of Eq. (1.5) and equalities (2.1), we obtain

$$b_z = \frac{d}{\rho(b_x + 2\Omega_z)} = \frac{db_x}{c}, \quad a_z = \frac{2\Omega_y b_x}{c}U(z) + \frac{hb_x}{c}; \quad U(z) = \int_0^z \rho b_x dz$$

The parameters c, d and h are constants of integration. Since the origin of coordinates $x = y = z = 0$ is assumed to be a fixed point of the lens, we can put the constants d and h equal to zero. Then

$$\upsilon_x = -b_x(z)y + a_z(z), \quad \upsilon_y = b_x(z)x$$

i.e. the quantity b_x has the meaning of the angular velocity $\omega(z)$ of points of the lens in the section $z = \text{const}$. The constant c, defined by formula (3.1), depends directly on the angular velocity, and hence is not equal to zero. From formulae (1.2) we can now obtain an explicit form of the partial derivatives of the pressure p with respect to the coordinates x, y and z, from a consideration of which it follows that the pressure inside the lens is given by the expression

$$\begin{gathered} p(x, y, z) = \frac{c}{2}(x^2 + y^2) - 2y\Omega_y U(z) - gW(z) + \frac{4\Omega_y^2}{c}\int_0^z \rho(z)b_x(z)U(z)dz + p_0 \\ W(z) = \int_0^z \rho(z)dz, \quad p_0 = p(0, 0, 0) \end{gathered} \tag{3.2}$$

4. The shape of the equilibrium surface

It follows from expression (3.2) that the equilibrium shape of the lens in each section $z = \text{const}$ is a circle with centre at the point with coordinates

$$x_0 = 0, \quad y_0 = 2\Omega_y U(z)/c \tag{4.1}$$

and radius given by the relation

$$\begin{aligned} \frac{R^2(z)c}{2} &= p_f(z) + \frac{2\Omega_y^2}{c}U^2(z) + gW(z) - p_0 - \frac{4\Omega_y^2}{c}\int_0^z \frac{dU(z)}{dz}U(z)dz = \\ &= p_f(z) + gW(z) - p_0 \end{aligned} \tag{4.2}$$

Here $p_f(z)$ is the pressure in the ocean at the corresponding depth. Naturally, the density ρ_f and the pressure p_f in the ocean are connected by the hydrostatics relation $dp_f/dz = -g\rho$. Formulae (4.1) and (4.2) are the basis of the proposed method of constructing the required equilibrium surface: this surface is regarded as the envelope of a family of circles of radius $R(z)$ in the corresponding sections $z = \text{const}$, where the centres of these circles form a certain line in the (y, z)

plane. The equilibrium shapes were constructed earlier in Refs. 1–3 as second-order surfaces in x, y, z coordinates in the special case of a uniform lens and a function ρ_f linear in z.

Expression (4.2) can be simplified further:

$$\frac{R^2(z)c}{2} = g\int_0^z [\rho(z)-\rho_f(z)]dz + p_{f_0} - p_0 \; (p_{f0} = p_f(0)) \tag{4.3}$$

whence, in particular, it follows that extremum values of R are observed at the "zero" level of z_ρ, where $\rho = \rho_f$. If, for the given value of z, the sign of the right-hand side of (4.3) is opposite to the sign of c, the plane z = const does not intersect the lens. Suppose, for simplicity, that $z_\rho = 0$. Then

$$R^2(z_\rho) = 2(p_{f_0} - p_0)/c \tag{4.4}$$

and, if this value is positive, the lens always has a certain equilibrium shape, even if it is thin and situated close to the section $z = 0$.

We will make the following assumption: a unique value of z_ρ exists such that

$$\rho > \rho_f \text{ when } z > z_\rho; \quad \rho < \rho_f \text{ when } z < z_\rho \tag{4.5}$$

This indicates that the upper (lower) layers of the lens contain a liquid of higher (correspondingly lower) density than the surrounding medium at this depth. Then, it follows from relation (4.3) that the lens can have a closed equilibrium shape only when $c < 0$. If condition (4.5) is satisfied and the equilibrium shape is closed, this means physically that the right-hand side of Eq. (4.4) is positive, i.e. the plane $z = 0$ intersects the lens and certain depths z correspond to such a lens of closed shape.

Using relation (4.3) we can verify that the following condition is satisfied

$$\int_{z_{\min}}^{z_{\max}} R^2(z)[\rho(z)-\rho_f(z)]dz = 0$$

which corresponds to Archimedes law. When $c < 0$ there are also non-closed equilibrium shapes, but only in the special case when the improper integrals $\int_0^{\pm\infty}[\rho(z) - \rho_f(z)]dz$ or at least one of these integrals converge.

If $c > 0$, expression (4.4) can only be positive when $p_{f0} > p_0$. An equilibrium shape also exists here but, in view of conditions (4.3) and (4.5), it is open.

According to equalities (3.1), $c = \rho\omega(\omega + 2\Omega_z)$ (ω is the angular velocity of natural rotation of the lens around the z axis in the section z = const), and we can also take the zero z level as the level of z at which to calculate this constant.

5. A lens in a multilayer medium

We will assume that the lens is situated in a medium consisting of three layers, which differ considerably in density. For simplicity suppose the densities of the upper and lower layers are constant. The upper, lighter, layer can be the atmosphere, while the lower, heavier, layer can either be a region of increased salinity or even the bottom of a reservoir, if the density of this layer approaches infinity. On contact with the lens the horizontal boundaries of the layers can easily be distorted.

When constructing the equilibrium shapes we will use formula (4.3) obtained above, which enables us to construct the shapes corresponding to each of the layers separately. The presence of several layers in the ocean undoubtedly is capable of somehow affecting the shape of anticyclones (more accurately, formations when $c < 0$), when this shape is closed. But this factor particularly strongly influences the equilibrium shape of cyclones (more accurately, formations with $c > 0$), by changing their open equilibrium shape to a closed shape.

Cyclones. Suppose condition (4.5) is satisfied. The equilibrium shape of cyclones in a single-layer medium resembles a one-sheet hyperboloid of infinite length with a curved axis along the line of the centres of rotation of the sections z = const. Any truncation of this formation with the separation of the central part, for example a consideration of the part between two planes perpendicular to the z axis, does not give equilibrium shapes. Only the whole formation can be regarded as an equilibrium shape, naturally, as a purely theoretical equilibrium shape. For certain distribution laws of the densities of the lens and the medium this formation may be split into two semi-infinite regions (here the zero level z_ρ is in a layer not belonging to the lens). These regions can also be regarded as a theoretical equilibrium shape, but only if they are also somehow connected with one another mechanically, since the lower part, which is lighter than the external medium, will tend to float up and it must be restrained from floating up by the upper part, which is heavier than the external medium.

In a three-layer (and even in a two-layer) external medium, the situation with these theoretical equilibrium shapes of the cyclones is capable of changing. According to relation (4.3), for positive c the derivative of the square of the radius R with respect to z is proportional to $g[\rho(z) - \rho_f(z)]$, i.e. the value of R^2 increases as z increases when $\rho > \rho_f$ (and the more rapidly the greater the difference in the densities) and decreases when $\rho < \rho_f$. When it comes in contact with the lighter (heavier) liquid the upper surface of the lens (correspondingly the lower surface) is deflected downwards (upwards), forming a lune; the tops of both lunes lie on the line of centres of rotation. When it comes in contact with the middle layer (the ocean) the lens forms the shape described in Section 4 – the analogue of a one-sheet hyperboloid, and its minimum section ("waist") arrives exactly at the level z_ρ. The slow change in the value of R^2 when it comes in contact with the middle layer is explained by the smallness of the modulus of the difference $\rho - \rho_f$; conversely, the relatively rapid change in R^2 in the lunes is due to the fact that this difference is not small. The closed equilibrium shape obtained also satisfies Archimedes law.

We will obtain the shape of the lune. Consider, for example, the upper lune. We transfer the origin of coordinates along the z axis to the bottom of the lune, where $R(z=0)=0$. We then have from Eq. (4.3)

$$R^2(z)c = 2g\left[\int_0^z \rho(z)dz - \rho_f^1 z\right] \approx 2gz[\rho(0) - \rho_f^1]$$

i.e. the shape of the lune is close to a paraboloid of revolution. Here ρ_f^1 is the density (constant, for simplicity) of the light layer of liquid. The depth of the lune (equal to the height h of the upper edge above the level $z=0$) is given by the approximate formula

$$h \approx \frac{R^2(h)c}{2g[\rho(0) - \rho_f^1]} \approx \frac{R^2(h)c}{2g[\rho(h) - \rho_f^1]} \tag{5.1}$$

Similar formulae can be obtained for the lower lune, and its depth is inversely proportional to $\rho_f^2 - \rho(-H)$, where H is the level of the lower surface of the middle layer and ρ_f^2 is the density of the heavy layer of liquid. In particular, where $\rho_f^2 \to \infty$ the depth of the lower lune approaches zero, i.e. the extremely dense lower layer is equivalent in its action to the bottom. An anticyclone with the same constant $c>0$ has the same equilibrium shape.

Cyclones can have real equilibrium shapes, i.e. bounded in space, in a two-layer medium,[3] in which, unlike the case considered above, there is no upper or no lower layer. If there is no lower layer, the lenses are heavy compared to the middle layer, and if they are lower, conversely, they are light. The excess or deficiency in the weight of a lens is compensated by the lunes, filled with liquid of light upper or heavy lower layers respectively. This enables the lens to be balanced in the middle layer.

In conclusion we will touch upon the case when the densities ρ and ρ_f increase with the depth of the ocean, but condition (4.5) is violated: there are several values of the zero level z_ρ. In this case, as follows from relation (4.3), the lens, even in a single-layer external medium, can also have a definite number of isolated closed vortices at different depths. In Section 6 we will discuss, in particular, some solutions relating to this type.

Anticyclones. In principle, all the discussions relating to the case of a cyclone also apply to anticyclones. The single important difference is the fact that anticyclones can have a closed equilibrium shape (of the ellipsoid type) in a single-layer external medium also. If other layers also play a role in forming this shape, "humps" rather than "lunes" occur on their boundary, the height of which is related to the radii of the base of the humps by relations similar to (5.1).

6. Numerical investigation

The aim of the numerical investigation of the problem was to construct the equilibrium shapes of the lenses, i.e., to construct specific relations between the radii of its horizontal sections R and the ocean depth z. We used actual

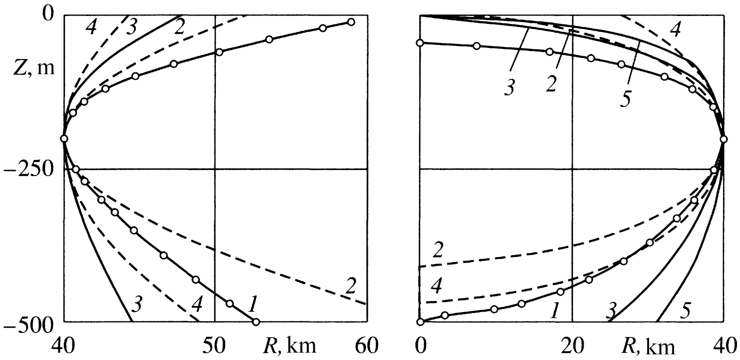

Fig. 1.

distributions of the ocean density over the depth for regions of the North Atlantic close to the coast of the USA.[6] The main calculations were carried out for the ocean in the neighbourhood of Cape Hatteras (latitude 35°N). We considered different versions of cyclonic ($c > 0$) and anticyclonic ($c < 0$) vortices, differing in the angular velocity of rotation ω at characteristic levels z_ρ; we considered versions of actual density distribution patterns in the ocean and lenses with different density gradients $d\rho/dz$. We assumed everywhere that at the level $z_{40} = -200$ m the value of R is exactly equal to 40 km. The fundamental formula (4.3) employed in these constructions enables the actual ocean density to be taken into account, since the necessary quadratures are easily determined using computer techniques. Some results of the numerical investigation are presented below.

In Fig. 1 we show graphs of $R = R(z)$ in the case of a cyclone (the left part of Fig. 1) for $\omega = 0.2\,\Omega$ (curves 1 and 2) and $\omega = 0.5\,\Omega$ (curves 3 and 4), and also in the case of an anticyclone (the right part of Fig. 1) for $\omega = -0.2\,\Omega$ (curves 1 and 2) and $\omega = -0.5\,\Omega$ (curves 3 and 4) respectively, when $z_\rho = z_{40}$ and $d\rho/dz = 0$. Graphs of $R(z)$ taking into account the actual ocean density in the so-called "western zone III-3",[6] are shown by the continuous curves, while the dashed curves show the same dependence but in the case when the ocean density is replaced by its linear representation. Mental rotation of any of the curves shown in Fig. 1 around the vertical z axis, corresponding to $R = 0$, reconstructs the qualitative pattern of the equilibrium surface of the lens. The equilibrium shapes in the case of a cyclone are open, while in the case of an anticyclone they are closed. It can be seen that the continuous curves, and the dashed curves corresponding to them, differ less the greater the modulus of ω.

The graphs give an approximate representation of the error in finding the equilibrium shapes when the actual ocean density is replaced by its linear approximation. Later the regime $z_\rho = z_{40}$, $d\rho/dz = 0$ and $\omega = 0.2\,\Omega$ for a cyclone and $\omega = -0.2\,\Omega$ for an anticyclone were considered as "basic", it was marked by points on the graph and different versions were constructed in the neighbourhood of this regime. In order to understand how any actual ocean density distribution pattern affects the equilibrium shape, we constructed graphs of $R(z)$ for the basic regime, where we took as the ocean density distribution laws the distributions not only in the III-3 zone but also in the "coastal I-1" and "Newfoundland I-2" zones.[6] (These zones are also close to Cape Hatteras.) In the case of both a cyclone and an anticyclone the graphs for zone I-1 hardly differs from curves 1. The graph for the I-2 zone in the case of a cyclone almost coincides with curve 3, although earlier when constructing it entirely different factors were taken into account. In the case of an anticyclone this is curve 5.

The most interesting results, perhaps, were obtained when specifying different values of $d\rho/dz \neq 0$. In Fig. 2 we show graphs in the neighbourhood of the basic regime of a cyclone. Curves 1–5 relate to the cases of an increase in the conventional density σ_t of the lens by 0, 0.1, 0.2, 0.3 and 0.46 units in each 100 m below the level z_{40}, respectively. (The conventional density $\sigma_t = 10^3 \times (\rho - 1\,\mathrm{g\,cm^{-3}})/1\,\mathrm{g\,cm^{-3}}$ is a dimensionless quantity and ρ is the density in $\mathrm{g\,cm^{-3}}$.) We draw attention to curve 5, which corresponds to an equilibrium shape closed from below with maximum depth of occurrence ($z = -500$ m). It consists of an open cyclonic top (half of a one-sheet hyperboloid) and a closed anticyclonic bottom (which recalls half of an ellipsoid in shape). The presence of such a combined shape can be explained by the more rapid growth of the density of the lens at deep levels compared with the increase in the ocean density. Thus, at a depth $z = -500$ m the conventional density is 27.66 for the lens and 26.75 for the ocean. However, the value 27.66 can occur in practice in the III-3 zone: an ocean density of this magnitude occurs at a depth of 2.5–3 km. (On the surface of the ocean in the III-3 zone the conventional density is approximately 24.5.)

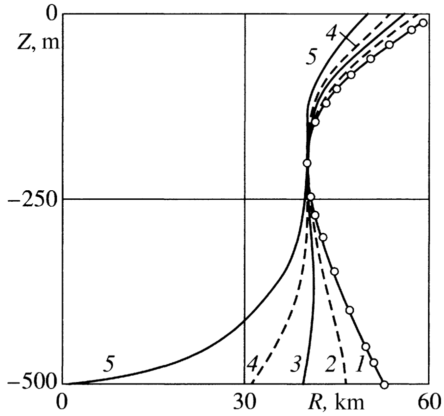

Fig. 2.

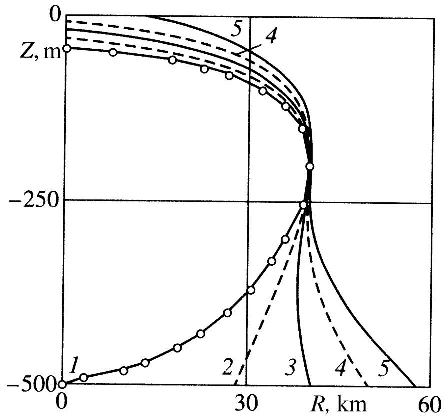

Fig. 3.

In Fig. 3 we show similar curves in the case of an anticyclone. Curves 1–5 relate to the cases when the conventional density σ_t of the lens increases by 0, 0.1, 0.2, 0.3 and 0.4 units in each 100 m below the level z_{40}. The equilibrium shapes here (apart from curve 1) consist of a closed anticyclonic top and an open cyclonic bottom. (In Fig. 2 the pattern was the opposite.) Here also curve 5 is the most characteristic. The increase in the value of R at deep levels is explained by the change in the sign of the constant c and the higher density of the material of the lens compared with the surrounding ocean.

The curves in Figs. 2 and 3 confirm that a change in the density inside the lens may be a factor capable of changing the closed form of the equilibrium shape into an open form and vice versa.

In Fig. 4 we show how the heights of the humps or the depths of the lunes (h, cm) depend on the radii R (km) of the vortices when they reach the ocean surface. The calculations were carried out using formula (5.1). The heights of the humps (positive values of h correspond to these) were determined for the basic anticyclone regime, and the depths of the lunes were calculated for the analogous case of a cyclone.

7. The equilibrium shapes of rotating lenses in the case of a vertical-radial distribution of their density

As a generalization of the initial formulation of the problem, we can consider the more general case of the density distribution of a vortex $\rho = \rho(z, r)$ and the same velocity field (1.4)

$$\upsilon_x = -\omega(z)[y - a_z(z)] = -\omega P, \ \upsilon_y = \omega(z)[x - b_z(z)] = \omega Q, \ \upsilon_z = 0; \ t^2 = P^2 + Q^2$$

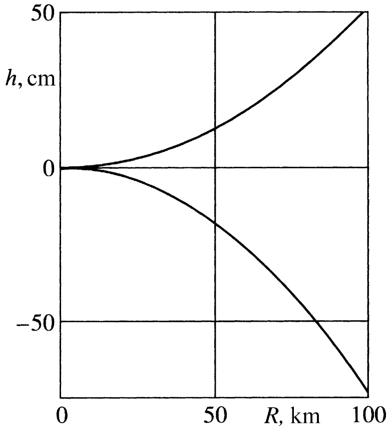

Fig. 4.

The continuity equations and the impermeability condition (1.8) are satisfied automatically. It can be shown that the pressure inside the lens will now be described by the formula

$$p(z, r) + \text{const} = -g\int_0^z \rho(\xi, 0)d\xi + \omega(z)[\omega(z) + 2\Omega_z]\int_0^r \eta\rho(z, \eta)d\eta$$

while the density must be related to the angular velocity by the relation

$$g\frac{\partial\rho}{\partial r} + r\frac{\partial\rho\omega(\omega + 2\Omega_z)}{\partial z} = 0$$

Referring to the case of actually existing lenses, where the angular velocity of natural rotation $\omega(z)$ is almost constant and equal to a certain quantity ω_0, we obtain from the last formula

$$\rho \approx F(z - r^2\omega_0[\omega_0 + 2\Omega_z]/(2g))$$

where F is a certain function. For actual values of ω_0 and for actual dimensions of the lenses the contribution of the term with the coefficient r^2 to the vertical coordinate z is small.

Acknowledgement

This research was supported financially by Russian Foundation for Basic Research (04-01-00387).

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Translated by R.C.G.